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A convergence for infinite dimensional vector valued functions

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Abstract By using the definition of Γ-convergence for vector valued functions given in Oppezzi and Rossi (Optimization, to appear), we obtain a property of infimum values of the Γ-limit. By generalizing Mosco convergence to vector valued functions, we also obtain, in the convex case, the extension of some stability results analogous to the ones in Oppezzi and Rossi (optimization, to appear), when domain and value space are infinite dimensional.

Keywords Vector optimization · Variational convergence · Stability of minima

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1 Introduction

In [12] we introduced a definition of convergence for sequences of functions whose values lie in a topological vector space partially ordered by a closed convex cone C. We call it Γ_C-convergence due to its analogy with the scalar case. For this type of convergence we proved a number of important general properties such as sequential characterization, lower semicontinuity of the Γ_C-limit and others well known in the scalar case (see [4] for an extensive treatment).

In the above mentioned paper we also obtained a variational property for a sequence (x_n) of ε_n-minimizers in the infinite dimensional case. However, in order to prove stronger results in the case of sequences of convex functions, analogous to the ones given in [10], we confined ourselves to finite dimensional spaces.

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In the general context of Γ_C-convergence here we give a unilateral variational property of infimum values of the Γ-limit in Theorem 3.3. Here we obtain an extension to the infinite dimensional case of some results which hold in the convex case, thanks to a stronger definition of convergence, which in the case of scalar functions reduces to the well known Mosco convergence.

As a preliminary result we get the strong continuity and the weak lower semicontinuity of C-convex functions, which are upper bounded on bounded sets.

In [11], Example 3.4, it was shown that if the ordering cone does not have a sequentially weakly compact base then Mosco convergence of sequences (A_n), even in separable Hilbert spaces, doesn't imply that minimum points in A_n approximate minimum points of the limit. So, in order to get that minimum values of the limit function f of the sequence (f_n) are strong limits of minimum values for f_n we assume the cone has a sequentially weakly compact base and the images of f_n satisfy the domination property.

Another difficulty due to the infinite dimension of the domain, arises in the proof of boundedness of sublevel sets. To address this we use the condition of recessive compactness. This sort of compactness was first introduced and discussed in [9] and in [8].

Moreover, in our framework we consider domains which vary with each element of the converging sequence (f_n). So, among preliminary results, we prove a lemma about the boundedness of the union of sublevel sets of f_n under a particular assumption on the sequence of domains, which is a generalization of recessive compactness to the case of a set sequence.

In this paper the assumption that the ordering cone has nonempty interior plays a crucial role. It is well-known that in the infinite dimensional case the nonnegative orthant of a great number of frequently used spaces has empty interior, so that our assumption appears rather restrictive. However it allows us to obtain several stability results analogous to the finite dimensional case.

2 Preliminaries and definitions

As a general assumption we shall denote by X a topological space and by Y a topological vector space endowed with a filter of neighborhoods of 0. More structural properties on X and Y will be precisely specified when necessary.

Definition 2.1 A set $C \subset Y$ is a pointed cone if

- $\lambda C \subset C, \quad \forall \lambda \geq 0$
- $(cl\, C) \cap (-cl\, C) = \{0\}$;

it is a convex pointed cone if additionally

- $C + C \subset C$.

Moreover we denote by $int\, C$ the interior of C and $C_o = C \setminus \{0\}$.

From now on we assume Y to be an ordered topological vector space with the partial order given as follows: $y,\ y' \in Y$, we write

$$y < y' \text{ iff } y' - y \in C_o,$$

$$y \leq y' \text{ iff } y' - y \in C,$$

where C is a pointed convex cone with $int\, C \neq \emptyset$.

Definition 2.2 (*see [13] Ch. 2, Proposition 1.3*) If Y is an ordered topological vector space with positive cone C and topology τ_Y, we say that C is normal with respect to τ_Y iff there exists a neighborhood basis W of the origin 0 consisting of sets V for which

$$0 \leq y' \leq y \in V \text{ implies } y' \in V.$$

Definition 2.3 If $A \subset Y$, $b \in Y$ is said to be a C-minorant for A iff $(b - C_o) \cap A = \emptyset$. We define

$$\mu_C(A) = \{b \in Y : b \text{ is a } C\text{-minorant for } A\}.$$

If $\mu_C(A) \neq \emptyset$, let

$$\inf_C A := \{b \in \mu_C(A) : (b + C_o) \cap \mu_C(A) = \emptyset\}.$$

If $A \neq \emptyset$, an element of the set

$$\min_C A := A \cap \inf_C A$$

is said to be a minimum for A or a Pareto (minimal) efficient point for A.
We call $b \in Y$ a weak C-minorant for A if

$$(b - int\, C) \cap A = \emptyset$$

and we denote by $W\mu_C(A)$ the set of all weak C-minorants of A.
Moreover, if $W\mu_C(A)$ is nonempty, we define the set of weak C-infima of A as:

$$W\inf_C A := \{b \in W\mu_C(A) : (b + int\, C) \cap W\mu_C(A) = \emptyset\}.$$

We say that $b \in A$ is a weak C-minimum for A or a weakly Pareto (minimal) efficient point for A, if $b \in W\mu_C(A)$ and we denote the set of such elements by $W\min_C\ A$.

Now we recall some well known concepts in vector optimization theory.

Definition 2.4

(i) If $f : X \to Y$, $\alpha \in Y$ we denote $f^\alpha = \{x \in X : f(x) \leq \alpha\}$.

(ii) (see e.g. [1]) Let Z be a vector space and $E \subset Z$. The recession cone of E is the set:

$$0^+(E) := \{d \in Z : a + td \in E\ \forall a \in E,\ \forall t \geq 0\}.$$

Definition 2.5 (*see [7] Ch. 1, Definition 6.1*) Let X be a topological vector space, $f : X \to Y$. We say that f is a C-convex function if for every $x_1, x_2 \in X$, $x_1 \neq x_2$ and for every $\lambda \in]0, 1[$ it results

$$f(\lambda x_1 + (1 - \lambda)x_2) \in \lambda f(x_1) + (1 - \lambda) f(x_2) - C.$$

We say that f is a strictly C-convex function if in the above condition C is replaced by $int\, C$.

Definition 2.6 Let $f : X \to Y$ and $E \subset X$. The set of efficient points of E for f is defined as

$$\text{Eff}\,(E, f) := \{x \in E : f(x) \in \min_C f(E)\}.$$

In [12] we introduced a definition of Γ_C-convergence for vector valued functions, which, when X is first countable, can be given as follows (see [12], Proposition 2.4).

Definition 2.7 Let $f_n, f\colon X \to Y$, $n \in \mathbb{N}$. We assume X to satisfy the first axiom of countability. We say that $(f_n)_{n\in\mathbb{N}}$ Γ_C-converges to f and we denote $f_n \overset{\Gamma_C}{\to} f$ iff

(a) For every $x \in X$ there exists in X a sequence $(x_n)_{n\in\mathbb{N}}$ converging to x such that $f_n(x_n) \to f(x)$;
(b) For every $x \in X$, $x_n \to x$, $\varepsilon \in int\, C$ there exists $k_{\varepsilon,x} \in \mathbb{N}$ such that $f_n(x_n) - f(x) + \varepsilon \in C$ for every $n \geq k_{\varepsilon,x}$.

We recall the notion of Mosco and Kuratowski convergence in the sequential form and we specify that if X is a topological vector space we adopt the usual notation $x_n \rightharpoonup x$, when the sequence $(x_n)_{n\in\mathbb{N}}$ in X weakly converges to $x \in X$.

Definition 2.8 Let X be a first countable topological vector space, $A_n, A \subset X$, we introduce the following notations:

$$s-\liminf A_n = \{x \in X : \exists\, x_n \to x,\ \text{such that}\ \ x_n \in A_n\ \forall n\}$$

$$s-\limsup A_n = \{x \in X : \exists\ (n_k)_{k\in\mathbb{N}}, \exists\, x_k \in A_{n_k}\ \text{such that}\ \ x_k \to x\}$$

$$w-\limsup A_n = \{x \in X : \exists\ (n_k)_{k\in\mathbb{N}}, \exists\, x_k \in A_{n_k}\ \text{such that}\ \ x_k \rightharpoonup x\}$$

and we say that

- the sequence $(A_n)_{n\in\mathbb{N}}$ converges to A in the sense of Mosco iff

$$s-\liminf A_n = w-\limsup A_n = A,$$

and we denote A by $M-\lim A_n$;

- the sequence $(A_n)_{n\in\mathbb{N}}$ converges to A in the sense of Kuratowski iff

$$s-\liminf A_n = s-\limsup A_n = A$$

and we denote A by $K-\lim A_n$.

Now we introduce a stronger kind of convergence with variable domains in order to obtain variational results analogous to the ones in [12] and [10] for the case of infinite dimensional space of values. Such convergence coincides with the well known Mosco convergence of functions in the case where the domain is fixed.

Definition 2.9 Let X be a first countable topological vector space, $f_n, f : X \to Y$, $A_n, A \subset X$, we say that the sequence $\left(f_{n|A_n}\right)$ is M_C-convergent to $f_{|A}$ ($f_{n|A_n} \overset{M_C}{\to} f_{|A}$) iff

- $A = M-\lim A_n$
- $\forall x \in A\ \exists\, (x_n)_{n\in\mathbb{N}},\ x_n \in A_n, x_n \to x$ and $f_n(x_n) \to f(x)$
- $\forall x \in A,\ x_n \rightharpoonup x,\ x_n \in A_n$ and every $\varepsilon \in int\, C\ \ \exists\, k_{\varepsilon,x} \in \mathbb{N}$ such that $f_n(x_n) - f(x) + \varepsilon \in C\ \ \ \forall n \geq k_{\varepsilon,x}$

3 Variational properties of Γ-convergence and stability in the convex case

We begin with a variational property which, under a suitable domination assumption, requires only the Γ_C-convergence.

Definition 3.1 We say that the *interior domination property* holds for a set $A \subset Y,\ A \neq \emptyset$, if

$$A \subset W\inf_C A + (int\, C \cup \{0\}).$$

Remark 3.2 We recall that a subset A of Y is said to satisfy the *domination property* (see e.g. [7]) iff for every $a \in A$ there exists $a' \in \min A$ such that $a \in a' + C$.

If $Y \supset A \neq \emptyset$, the interior domination property always holds when Y is finite dimensional (see [14] Corollary 3.1), so it is weaker than the domination property.

Theorem 3.3 *Let X be a topological space and Y a topological vector space ordered by a closed convex pointed cone C with $int\, C \neq \emptyset$ and $f_n,\ f : X \to Y,\ f = \Gamma_C - \lim f_n$.*

We assume that $W\inf_C f_n(X) \neq \emptyset$ and that the interior domination property holds for $f_n(X)$ for every $n \in \mathbb{N}$. Then for each $\bar{y} \in W\inf_C f(X)$, for each $\varepsilon \in int\, C$ there exists $n_\varepsilon \in \mathbb{N}$ with the property

$$\forall\, n \geq n_\varepsilon\ \exists\ y_n \in W\inf_C f_n(X),\ \ y_n < \bar{y} + \varepsilon.$$

Proof From Definition 2.3 there exists $x \in X$ such that $f(x) < \bar{y} + \dfrac{\varepsilon}{2}$.
By Definition 2.7 there exists a sequence $(x_n)_{n\in\mathbb{N}}$ converging to x and $n_\varepsilon \in \mathbb{N}$ such that

$$\forall\, n \geq n_\varepsilon \quad f(x) + \frac{\varepsilon}{2} > f_n(x_n).$$

Due to the interior domination property there exist $y_n \in W\inf_C f_n(X)$ and $\varepsilon' \in int\, C \cup \{0\}$ satisfying $f_n(x_n) = y_n + \varepsilon'$.
Hence $y_n \leq f_n(x_n) < f(x) + \dfrac{\varepsilon}{2} < \bar{y} + \varepsilon$. □

From now on we assume X, Y to be normed spaces.
In the sequel we need a result on continuity for convex functions which is based on the following definition.

Definition 3.4 We say that $f : X \to Y$ is locally upper bounded iff for any $x \in X$ there exists a bounded neighborhood U of x and there exists $y_U \in Y$ such that

$$y_U - f(x') \in C \quad \forall\, x' \in U.$$

Theorem 3.5 *Let C be a closed convex pointed cone which is normal with respect to the norm topology. If $f : X \to Y$ is a C-convex locally upper bounded function, then f is continuous.*

Proof The proof is analogous to the one of [5] Lemma 2.1, Ch. 1. We give it for sake of completeness. Without loss of generality we can prove the continuity of f in $\bar{x} = 0$ assuming also $f(0) = 0$. Let V be a ball of center in the origin and $y_o \in Y$ such that

$$f(x) \leq y_o \quad \forall\ x \in V$$

We consider $\varepsilon \in \mathbb{R},\ 0 < \varepsilon < 1$ and $x \in \varepsilon V$. Since $\dfrac{x}{\varepsilon},\ -\dfrac{x}{\varepsilon} \in V$, due to C-convexity of f we get

$$f(x) \leq (1-\varepsilon) f(0) + \varepsilon f(\frac{x}{\varepsilon}) = \varepsilon f(\frac{x}{\varepsilon}) \leq \varepsilon y_o$$

$$0 = f(0) \leq \frac{1}{1+\varepsilon} f(x) + \frac{\varepsilon}{1+\varepsilon} f(-\frac{x}{\varepsilon}),$$

and so $f(x) \geq -\varepsilon f(-\dfrac{x}{\varepsilon}) \geq -\varepsilon y_o$.

Then we have obtained

$$-\varepsilon y_o \leq f(x) \leq \varepsilon y_o \quad \forall\, x \in \varepsilon V.$$

Since C is normal by [13], Proposition 1.7 Ch. 2, there exists a constant $\gamma > 0$ such that $\|f(x) + \varepsilon y_o\| \leq \gamma 2\varepsilon \|y_o\|$.
Therefore $\|f(x) - f(0)\| \leq (1 + 2\gamma)\varepsilon\|y_o\|$ which implies the continuity of f in 0. □

Corollary 3.6 *Let C be a closed convex normal (with respect to the norm topology) cone. If $f : X \to Y$ is C-convex and locally upper bounded, then f is sequentially lower$_C$-semicontinuous with respect to the weak convergence (see [3]), i.e.*

$$f^{-1}(y - C) \text{ is sequentially closed in the weak topology } \forall\, y \in Y.$$

Proof Clearly $f^{-1}(y-C)$ is convex and strongly closed thanks to continuity of f (Theorem 3.5). Hence the assertion follows from Mazur's Lemma. □

Proposition 3.7 *Let $E \subset X$ be a convex set and $f : X \to Y$ a C-convex function. Then*

(a) *$f(E) + C$ is a convex subset of Y;*
(b) *if f is strictly C-convex then*

$$\{x \in E : f(x) \in W\min_C f(E)\} = \mathrm{Eff}(E, f).$$

Proof The assertion a) is obvious and b) is proved in [7] Ch. 2, Proposition 5.13. □

Proposition 3.8 *Let C be a closed convex pointed cone which is normal with respect to the norm topology. Let $f : X \to Y$ be a C-convex locally upper bounded function. If $a,\ b \in Y$ and $f^a \neq \emptyset,\ f^b \neq \emptyset$ then $0^+(f^a) = 0^+(f^b)$.*

Proof Thanks to Theorem 3.5 the proof is as the one of [10], Proposition 2.2. □

In virtue of the above Proposition we may denote by H_f the recession cone of any nonempty sublevel set of a C-convex function.

We give now a definition which, as we observe in the next remark, in the case of a normed space, is analogous to the one given by Luc in [8] Def. 2.1 for the general case of Hausdorff topological vector spaces.

Definition 3.9 A set $A \subset X$ is said to be recessively compact if for every unbounded sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A there exist a subsequence (a_{n_k}), a sequence (t_k) of positive numbers, $t_k \to 0$ and $\bar{a} \in X \setminus \{0\}$ such that $t_k\, a_{n_k} \rightharpoonup \bar{a}$.

Remark 3.10 We recall to the reader the original definition given by Luc in [8].
" Let (X, θ) be a real Hausdorff topological vector space.

A nonempty set $A \subseteq X$ is said to be recessively compact (or r-compact) if for every ρ-unbounded net $\{a_\alpha\}_{\alpha\in I} \subseteq A$ there is a subnet $\{a_\beta\}_{\beta\in I'} \subseteq \{a_\alpha\}_{\alpha\in I}$ and positive numbers t_β converging to 0 such that $\{t_\beta a_\beta\}_{\beta\in I'}$ converges to some nonzero limit."

We observe that such definition coincides with ours if X is a normed space and $\theta = w$ is the weak topology.

In fact in the case of a normed space an unbounded sequence $(a_n)_{n\in\mathbb{N}}$ is also ρ-unbounded as in Definition 2.1 of [8], which means that there exists a neighborhood U of the origin such that $\limsup \rho_U(a_n) = +\infty$, where $\rho_U(x) = \inf\{t > 0\colon\ x \in tU\}$.

To see this, assume $(a_n)_{n\in\mathbb{N}}$ to be norm unbounded; if it were bounded in the weak topology, by [6] §15, n.6- (3) it would follow that $\alpha_n\, a_n \rightharpoonup 0$ whenever $\alpha_n \to 0$. But this is impossible if we choose $\alpha_n = \dfrac{1}{\sqrt{\|a_n\|}}$.

Sufficient conditions for recessive compactness are given in [8] Proposition 2.2.

Proposition 3.11 *Let $E \subset X$ be a closed convex recessively compact set, $f : X \to Y$ a locally upper bounded C-convex function. The following assertions are equivalent:*

(i) $0^+(E) \cap H_f = \{0\}$;
(ii) *$f^a \cap E$ is bounded for every $a \in Y$.*

Proof It is clear that (i) is a consequence of (ii).
Conversely by contradiction we assume the existence of a sequence $x_n \in f^a \cap E$, $\|x_n\| \to \infty$. Due to recessive compactness of E, we may consider $\bar{x} \in X \setminus \{0\}$ and a sequence $(t_k)_{k\in\mathbb{N}}$ in $\mathbb{R}_+$, $t_k \to 0$ such that $t_k x_{n_k} \rightharpoonup \bar{x}$, where (x_{n_k}) is a subsequence of (x_n).

We prove that in such a case $\bar{x} \in 0^+(E) \cap H_f$. In fact if $x \in f^a \cap E$ it results, when $t \geq 0$ and k is sufficiently large, $(1 - t\, t_k)x + t\, t_k\, x_{n_k} \in f^a \cap E$. On the other hand $(1 - t\, t_k)x + t\, t_k\, x_{n_k} \rightharpoonup x + t\,\bar{x}$ and, by continuity and convexity of f, the set f^a is closed. Therefore $x + t\,\bar{x} \in E \cap f^a$, which concludes the proof, because $0^+(E \cap f^\alpha) \subset 0^+(E) \cap H_f$. □

In order to study M_C-convergent sequences we give the following definition which coincides with Definition 3.9 in the case of a single set.

Definition 3.12 Let $(E_k)_{k\in\mathbb{N}}$ be a sequence of subsets of X. We say that $(E_k)_{k\in\mathbb{N}}$ is a recessively compact sequence iff for every subsequence $(E_{n_k})_{k\in\mathbb{N}}$ and every norm unbounded sequence $(x_k)_{k\in\mathbb{N}}$, $x_k \in E_{n_k}$ $\forall k \in \mathbb{N}$ there exist a further subsequence $(x_{k_j})_{j\in\mathbb{N}}$, a sequence (t_j) of positive numbers, $t_j \to 0$ and $\bar{x} \in X \setminus \{0\}$ such that $t_j\, x_{k_j} \rightharpoonup \bar{x}$.

Remark 3.13 The following assertions are straightforward.

- If X is a normed space, $(E_k)_{k\in\mathbb{N}}$ is a recessively compact sequence and $X \supset E = M - \lim E_k$, then E is recessively compact.
- A sufficient condition for $(E_k)_{k\in\mathbb{N}}$ to be a recessively compact sequence is that the set $\bigcup_{k\geq\bar{k}} E_k$ is recessively compact for some $\bar{k} \in \mathbb{N}$.

Lemma 3.14 *Let E_n, $E \subset X$ be closed convex sets, $(E_n)_{n\in\mathbb{N}}$ a recessively compact sequence and $E = M - \lim E_n$. Let $f_n, f : X \to Y$ be C-convex locally upper bounded functions such that $f_{n|E_n} \overset{M_C}{\to} f_{|E}$. Moreover we assume $0^+(E) \cap H_f = \{0\}$.*
Then $0^+(E_n) \cap H_{f_n} = \{0\}$ if n is sufficiently large.

Proof By contradiction if the assertion is false, there exists a subsequence $d_k \in 0^+(E_{n_k}) \cap H_{f_{n_k}}$, $\|d_k\| \to +\infty$. If $a \in E$ is given, let $a_k \in E_{n_k}$, $k \in \mathbb{N}$ be such that $a_k \to a$. Since $a_k + d_k \in E_{n_k}$, thanks to recessive compactness, possibly for a subsequence, there exist $\alpha_k \to 0$, $\alpha_k > 0$ and $\bar{x} \in X \setminus \{0\}$ such that $\alpha_k(a_k + d_k) \rightharpoonup \bar{x}$. Then, if $t \in \mathbb{R}_+$ is given, we have $a_k + \alpha_k\, t\, d_k \rightharpoonup a + t\,\bar{x}$ and having $a_k + \alpha_k\, t\, d_k \in E_{n_k}$, it follows that $a + t\,\bar{x} \in E$. Now, by arbitrariness of a and t we get $\bar{x} \in 0^+(E)$.

Let us prove that $\bar{x} \in H_f$.

By Definition 2.9, if $x \in E$ there exist $x_k \in E_{n_k}$, $x_k \to x$ such that $f_{n_k}(x_k) \to f(x)$. Let $\alpha = f(x)$, $\varepsilon \in int\, C$, if α_k and d_k are chosen as before, in virtue of Proposition 3.8 it follows that $d_k \in 0^+(f_{n_k}^{\alpha+\varepsilon})$, so $x_k + \alpha_k\, t\, d_k \in f_{n_k}^{\alpha+\varepsilon}$ for large $k \in \mathbb{N}$.

On the other hand we have $x_k + \alpha_k\, t\, d_k \rightharpoonup x + t\,\bar{x}$ and, by M_C-convergence hypothesis,

$$f(x + t\,\bar{x}) - \varepsilon < f_{n_k}(x_k + \alpha_k\, t\, d_k) < \alpha + \varepsilon$$

for k large enough. By arbitrariness of ε we conclude that $\bar{x} \in 0^+(f^\alpha)$ □

Lemma 3.15 *Under the assumptions of Lemma 3.14, if $\alpha \in Y$ is such that $f^\alpha \cap E \neq \emptyset$ there exists $\bar{n} \in \mathbb{N}$ such that $\bigcup_{n \geq \bar{n}} (E_n \cap f_n^\alpha)$ is bounded.*

Proof By contradiction we suppose $\bigcup_{n \geq \bar{n}} (E_n \cap f_n^\alpha)$ to be unbounded for any $\bar{n} \in \mathbb{N}$. Then there exists an unbounded sequence $x_k \in E_{n_k} \cap f_{n_k}^\alpha$ and by recessive compactness of the sequence (E_n), for a further subsequence, there exist $\bar{x} \neq 0,\ \alpha_k \in \mathbb{R}_+,\ \alpha_k \to 0$ such that $\alpha_k x_k \rightharpoonup \bar{x}$. Let's now consider $x' \in E,\ x'_k \in E_{n_k}$ such that $x'_k \to x'$ and $f_{n_k}(x'_k) \to f(x')$. Then, if $t > 0$, we obtain:

$$(1 - t\alpha_k)x'_k + t\alpha_k x_k \rightharpoonup x' + t\bar{x},$$

so $\bar{x} \in 0^+(E)$.
Moreover we take $x \in f^\alpha \cap E,\ x''_k \in E_{n_k},\ x''_k \to x$ such that $f_{n_k}(x''_k) \to f(x)$; then

$$(1 - t\alpha_k)x''_k + t\alpha_k x_k \rightharpoonup x + t\bar{x}.$$

Thanks to M_C-convergence for each $\varepsilon \in int\, C$ there exists k_ε such that

$$\begin{aligned} f(x + t\bar{x}) - \varepsilon &\leq f_{n_k}((1 - t\alpha_k)x''_k + t\alpha_k x_k) \\ &\leq (1 - t\alpha_k) f_{n_k}(x''_k) + t\alpha_k f_{n_k}(x_k) \leq (1 - t\alpha_k) f_{n_k}(x''_k) + t\alpha_k \alpha. \end{aligned}$$

The right hand side in the last inequality converges to $f(x)$ which is majorized by α, so, by arbitrariness of ε it follows that $x + t\bar{x} \in f^\alpha$. Then we conclude that

$$x + t\bar{x} \in f^\alpha \cap E \qquad \forall\, x \in f^\alpha \cap E, \forall\, t > 0$$

which implies the unboundedness of $f^\alpha \cap E$ in contradiction with Proposition 3.11. □

Lemma 3.16 *Let $A \subset Y$ be a bounded set and C a cone in Y such that $int\, C \neq \emptyset$. Then there exists $\bar{y} \in C$ such that $\bar{y} - A \subset C$.*

Proof Let $\varepsilon \in int\, C$ and $B(\varepsilon, \delta) \subset C$. If $r \in \mathbb{R}_+$ and $B(0, r) \supset A$, we choose $\bar{y} = \lambda\varepsilon$ with $\lambda > r/\delta$ in order to have $\{\varepsilon + \frac{x}{\lambda} \ :\ x \in B(0, r)\} \subset B(\varepsilon, \delta)$. Hence $B(\bar{y}, r) \subset \lambda B(\varepsilon, \delta) \subset C$ and consequently $-C \supset -B(\bar{y}, r) = B(-\bar{y}, r) = B(0, r) - \bar{y}$. Then $\bar{y} - A \subset C$. □

Theorem 3.17 *Let $E_n,\ \ E \subset X$ be closed convex sets, $(E_n)_{n\in\mathbb{N}}$ a recessively compact sequence and $E = M - \lim E_n$. Let $f_n, f : X \to Y$ be C-convex locally upper bounded functions such that $f_{n|E_n} \overset{M_C}{\to} f_{|E}$. Moreover we assume $0^+(E) \cap H_f = \{0\}$.
Then $f(E) + C = M - \lim(f_n(E_n) + C)$.*

Proof We begin to prove the inclusion $f(E) + C \subset s - \liminf(f_n(E_n) + C)$.

Let $y \in f(E) + C$ and $x \in E$ such that $f(x) \leq y$. By definition of M_C-convergence there exists a sequence $(x_n)_{n\in\mathbb{N}},\ x_n \in E_n,\ x_n \to x$ such that $f_n(x_n) \to f(x)$.

Then if $\varepsilon \in int\, C$, there exists $k_\varepsilon \in \mathbb{N}$ such that $f_n(x_n) < y + \varepsilon \quad \forall n \geq k_\varepsilon$, so it follows that $y + \varepsilon \in f_n(E_n) + C \quad \forall\, n \geq k_\varepsilon$.

Hence $y + \varepsilon \in s - \liminf(f_n(E_n) + C)$, but this set is closed and, by arbitrariness of ε, it results $y \in s - \liminf(f_n(E_n) + C)$.
Now let us prove the inclusion $w - \limsup(f_n(E_n) + C) \subset f(E) + C$.

We consider $y \in Y$ such that for a suitable subsequence $(n_k)_{k\in\mathbb{N}}$ there exists $y_k \in f_{n_k}(E_{n_k}) + C,\ y_k \rightharpoonup y$. By previous lemma there exists $\alpha \in C$ such that $y_k \leq \alpha$ for every

$k \in \mathbb{N}$. If we take $x_k \in E_{n_k}$ such that $f_{n_k}(x_k) \leq y_k$ we obtain $x_k \in E_{n_k} \cap f^{\alpha}_{n_k}$ for each $k \in \mathbb{N}$. Thanks to Lemma 3.15 the sequence (x_k) is bounded, hence at least for a subsequence, we get $x_k \rightharpoonup x \in E$. In virtue of Definition 2.9 for every $\varepsilon \in int\, C$ there exists $h_\varepsilon \in \mathbb{N}$ such that

$$f(x) - \varepsilon < f_{n_k}(x_k) \leq y_k \quad \forall k > h_\varepsilon.$$

By closedness and convexity of C it results

$$y - f(x) + \varepsilon = \lim(y_k - f(x) + \varepsilon) \in C$$

and by arbitrariness of ε we conclude that $y - f(x) \in C$, i.e. $y \in f(E) + C$. □

Lemma 3.18 *Let Y be a reflexive Banach space, C a closed convex normal (with respect to the norm topology) cone. Let $E \subset X$ be a closed and convex set, $f : X \to Y$ be C-convex and locally upper bounded. Moreover we assume that $0^+(E) \cap H_f = \{0\}$.*
Then $f(E) + C$ is closed.

Proof Thanks to Theorem 3.6 and Proposition 3.11 the proof is analogous to the one of Lemma 2.1 in [10]. □

Theorem 3.19 *Under the hypotheses of Lemma 3.14, if we assume that:*

- *C has a sequentially weakly compact base,*
- *$f_n(E_n)$ satisfies the domination property for every $n \in \mathbb{N}$, then* $\min f(E) \subset s-\liminf(\min f_n(E_n))$.

Proof By Lemma 3.14 we have $0^+(E_n) \cap H_{f_n} = \{0\}$ for large n. Then Lemma 3.18 ensures that $f(E) + C$ and $f_n(E_n) + C$ are closed. Clearly $\min(I + C) = \min I$ for every subset $I \subset Y$ and if $f_n(E_n)$ satisfies the domination property the same holds for $f_n(E_n) + C$. In virtue of Theorem 3.17 we can apply Theorem 3.1 of [11], obtaining that

$$\min(f(E) + C) \subset s - \liminf \min(f_n(E_n) + C),$$

which concludes the proof. □

Remark 3.20 Under the assumptions of Lemma 3.14, if C has a sequentially weakly compact base, a sufficient condition for the domination property of $f_n(E_n)$ is the following:

$$\forall\, n \in \mathbb{N}\ \exists\, y_n \in Y \text{ such that } f_n(E_n) \subset y_n + C.$$

In fact thanks to Lemma 3.18, $f_n(E_n)+C$ is closed and "minorized" (i.e. there exists $\bar{y}_n \in Y$ such that $f_n(E_n) + C \subset \bar{y}_n + C$), hence each "section" $(f_n(E_n) + C)_y := (f_n(E_n) + C) \cap (y - C)$ is closed and minorized. Moreover by [2], Prop. 3.4, C is a Daniell cone because it has a weakly sequentially compact base. Due to Lemma 3.5, Ch. 3 of [7] then $(f_n(E_n)+C)_y$ is C-complete and Theorem 4.3 of [7] ensures the domination property.

Theorem 3.21 *Under the assumptions of Lemma 3.14 it results*

$$s - \limsup\,(W\min f_n(E_n)) \subset W\min f(E)$$

Proof Let $y_n \to y,\ y_n \in W\min f_n(E_n)$, we prove that $y - int\, C \cap f(E) = \emptyset$.
By contradiction, if $\varepsilon \in int\, C$ and $y - \varepsilon = f(x),\ x \in E$, by M_C-convergence there exists a sequence $(x_n)_{n\in\mathbb{N}}$ strongly convergent to $x,\ x_n \in E_n$ such that $f_n(x_n) \to f(x)$.
Therefore we can take $\bar{n}_\varepsilon \in \mathbb{N}$ such that

$$f_n(x_n) < y - \frac{\varepsilon}{4}, \quad y - \frac{\varepsilon}{8} < y_n \quad \forall\, n > \bar{n}_\varepsilon.$$

This gives a contradiction, because it implies $f_n(x_n) \in y_n - int\, C$. □

Theorem 3.22 *In addition to the assumptions of Theorem 3.19, we also suppose f to be strictly C-convex. Then*

$$\min_C f(E) = K - \lim \min_C f_n(E_n)$$

Proof It clearly follows from Theorem 3.19 and Theorem 3.21, because by strict convexity $\min f(E) = W \min f(E)$. □

Theorem 3.23 *Under the hypotheses of the previous theorem it follows that*

$$\text{Eff}(E, f) = M - \lim \text{Eff}\,(E_n, f_n)$$

Proof Thanks to Lemma 3.15 and definition of M_C-convergence the proof is the same as that of Theorem 4.15 in [12]. □

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